JOURNAL OF APPROXIMATION THEORY **1**, 251–254 (1968)

On the Definition of Restricted Infrapolynomials[1]

MISHAEL ZEDEK

Department of Mathematics, University of Maryland, College Park, Maryland 20742
Communicated by J. L. Walsh

1. Let E be a compact point-set in the complex z-plane containing at least $n(\geqslant 1)$ points and let P_n be the class of polynomials $z^n + \sum_{i=1}^{n} c_i z^{n-i}$ with complex coefficients. Fekete and von Neumann [*1*] introduced *infrapolynomials* on E as those polynomials in P_n which have no *underpolynomials* in P_n on E, namely,

DEFINITION 1. $p(z) \in P_n$ is called an infrapolynomial on E, if there is no other polynomial $q(z) \in P_n$ which satisfies

$$|q(z)| < |p(z)| \quad \text{for } z \in E \text{ where } p(z) \neq 0, \tag{1}$$

$$q(z) = p(z) = 0 \quad \text{for } z \in E \text{ where } p(z) = 0. \tag{2}$$

Motzkin and Walsh [*4*] have shown that infrapolynomials on E have no *weak* underpolynomials on E, and more specifically

THEOREM 1. *$p(z) \in P_n$ is an infrapolynomial on E if and only if there is no other polynomial $q(z) \in P_n$ which satisfies*

$$|q(z)| \leqslant |p(z)| \quad \textit{for all } z \in E. \tag{3}$$

Clearly the above theorem may serve as a more satisfactory definition of infrapolynomials, which can also be phrased as follows:

DEFINITION 1*. $p(z) \in P_n$ is called an infrapolynomial on E if for each other polynomial $q(z) \in P_n$ there is a point $z_q \in E$ such that

$$|q(z_q)| > |p(z_q)|. \tag{4}$$

It is the purpose of this note to show that this simplification can be carried over to some classes of *restricted* polynomials but not to others since an analogous theorem to Theorem 1 does not generally hold true.

2. Restricted infrapolynomials were introduced by various authors (see Zolotarev [*8*], Walsh and Zedek [*7*], Meĭman [*3*], Shisha and Walsh [*5*], and

[1] This research was supported by the National Science Foundation Grant GP-6891 to the University of Maryland.

Walsh [6]). They differ from the infrapolynomials defined above by the fact that both the restricted infrapolynomials and the underpolynomials belong to a subclass $P_n(A_{n_1},\ldots,A_{n_q})$ of P_n consisting of all polynomials of the form $z^n+\sum_{i=1}^n c_i z^{n-i}$, with $q(<n)$ prescribed coefficients $c_{n_k}=A_{n_k}$, $k=1,\ldots,q$.

DEFINITION 2. $p(z)\in P_n(A_{n_1},\ldots,A_{n_q})$ is called a restricted infrapolynomial on E if there is no other polynomial $q(z)\in P_n(A_{n_1},\ldots,A_{n_q})$ which satisfies

$$|q(z)|<|p(z)| \quad \text{for } z\in E \text{ where } p(z)\neq 0, \tag{5}$$

$$|q(z)|=|p(z)| \quad \text{for } z\in E \text{ where } p(z)=0. \tag{6}$$

Restricted infrapolynomials may, in general, have *weak* underpolynomials on E (i.e., a polynomial $q(z)$ in the same subclass as the restricted infrapolynomial $p(z)$, satisfying $q(z)\not\equiv p(z)$, $|q(z)|\leqslant|p(z)|$ for all $z\in E$) even though they cannot have underpolynomials on E, by Definition 2. The following example will illustrate this situation:

Let $E=\{-2,2\}$ and consider for $A_2=0$, the subclass $P_3(A_2)$, of polynomials of the form $p(z)=z^3+a_1z^2+a_3$. We shall see that all the polynomials of the above form whose coefficients satisfy

$$-8\leqslant 4a_1+a_3\leqslant 8 \tag{7}$$

are restricted infrapolynomials on E. Indeed, let $q(z)=z^3+b_1z^2+b_3$. Then

$$\begin{aligned}|q(2)|+|q(-2)|\geqslant q(2)-q(-2)=16&=8-|4a_1+a_3|+8+|4a_1+a_3|\\ &=|p(2)|+|p(-2)|,\end{aligned}$$

the last equality following from (7). Clearly, $|q(2)|<|p(2)|$ implies $|q(-2)|>|p(-2)|$, $|q(-2)|<|p(-2)|$ implies $|q(2)|>|p(2)|$, and we cannot have $|q(2)|=|p(2)|=|q(-2)|=|p(-2)|=0$ unless $q(z)\equiv p(z)$. Thus $p(z)$ has no underpolynomials on E. On the other hand, $r(z)=z^3+(a_1-1)z^2+(a_3+4)$ is a weak underpolynomial to $p(z)$ on E, since $p(-2)=r(-2)$ and $p(2)=r(2)$.

3. The example demonstrates that it is impossible, in general, to replace (5) and (6) in Definition 2 by a single inequality $|q(z)|\leqslant|p(z)|$ for all $z\in E$. Nevertheless, in an important case, this can be done. That is the case, significant for the approximation of polynomials of a high degree by polynomials of a lower degree, of prescribed *consecutive* leading coefficients.

We adapt Motzkin and Walsh's proof of Theorem 1 above, to prove

THEOREM 2. *$p(z)\in P_n(A_1,\ldots,A_s)$ is a restricted infrapolynomial on the compact set E (with respect to the subclass $P_n(A_1,\ldots,A_s)$ of polynomials of the form $z^n+\sum_{i=1}^s A_iz^{n-i}+\sum_{i=s+1}^n a_iz^{n-i}$) if and only if there is no other polynomial $r(z)\in P_n(A_1,\ldots,A_s)$ which satisfies*

$$|r(z)|\leqslant|p(z)| \quad \textit{for all } z\in E. \tag{8}$$

Proof. The "if" part of the proposition is trivial. To prove the "only if" part, let us assume that $p(z)$ is a restricted infrapolynomial on E and let us show that if there exists a weak underpolynomial $r(z)$ satisfying (8), then one can construct an underpolynomial $q(z) \in P_n(A_1, \ldots, A_s)$ satisfying (5) and (6), contradicting our assumption about $p(z)$.

It is clear that $m(z) = [r(z) + p(z)]/2$ belongs to $P_n(A_1, \ldots, A_s)$ and satisfies in E either $|m(z)| < |p(z)|$ or $m(z) = p(z)$. The later equality can hold in at most $n - s - 1$ points of E (counting multiplicities), since otherwise $m(z) \equiv p(z) \equiv r(z)$. Some of these, say $z_1, \ldots, z_k$, $0 \leqslant k \leqslant n - s - 1$, may be common zeros. Assume $p(z) = p_1(z) f(z)$ and $m(z) = m_1(z) f(z)$ where

$$f(z) = \textstyle\prod_{i=1}^{k} (z - z_i) = z^k + B_1 z^{k-1} + \ldots + B_k$$

(this step in the proof should be omitted if $k = 0$). We note that the s leading coefficients $u_1, u_2, \ldots, u_s$ in both $p_1(z)$ and $m_1(z)$ are identical, since they must satisfy the equations

$$\left.\begin{aligned} A_1 &= B_1 + u_1 \\ A_2 &= B_2 + B_1 u_1 + u_2 \\ &\ldots \\ A_s &= B_s + B_{s-1} u_1 + \ldots + B_1 u_{s-1} + u_s \end{aligned}\right\} \tag{9}$$

(where $B_i = 0$ for $i > k$).

Suppose now that $m(z) = p(z) \neq 0$ on t points of E. Clearly

$$0 \leqslant t \leqslant n - s - 1 - k,$$

but the case $t = 0$ is trivial, since then, $m(z)$ itself would be an underpolynomial of $p(z)$ on E. Assume, then, that $S = \{w_1, \ldots, w_t\}$ is the set of points in E where $m(z) = p(z) \neq 0$, and thus also, where $m_1(z) = p_1(z) \neq 0$. We have

$$0 < |m_1(z)| < |p_1(z)| \quad \text{for } z \in E - S. \tag{10}$$

Let $L(z)$ be the Lagrange polynomial of interpolation of degree $t (\leqslant n - s - 1 - k)$ satisfying

$$L(z) = p_1(z) = m_1(z) \quad \text{for } z \in S. \tag{11}$$

Since $0 = |m_1(z) - L(z)| < |p_1(z)|$ for $z \in S$, the same inequality would hold for an open neighbourhood U of S. Hence, we have for all e, $0 < e < 1$,

$$|e[m_1(z) - L(z)] + (1 - e) m_1(z)| = |m_1(z) - eL(z)| < |p_1(z)|, \quad \text{for } z \in U. \tag{12}$$

On the compact set $E - U$ we have $|m_1(z)| < |p_1(z)|$, and if e is sufficiently small,

$$|m_1(z) - eL(z)| < |p_1(z)| \quad \text{for } z \in E - U. \tag{13}$$

It remains to point out that the polynomial

$$m_1(z) - eL(z) = z^{n-k} + u_1 z^{n-k-1} + \ldots + u_s z^{n-k-s} + \ldots$$

has the same leading coefficients as $p_1(z)$, and due to equations (9), the polynomial $q(z) = [m_1(z) - eL(z)]f(z)$ belongs to $P_n(A_1, \ldots, A_s)$. Moreover, $|q(z)| < |p(z)|$ everywhere in E except at their common zeros $z_1, \ldots, z_k$ (if any). This completes the proof of Theorem 2, which permits us to give an alternate definition to restricted infrapolynomials having prescribed consecutive leading coefficients:

DEFINITION 3. $p(z) \in P_n(A_1, \ldots, A_s)$ is called a restricted infrapolynomial on E if for each other polynomial $q(z) \in P_n(A_1, \ldots, A_s)$ there is a point $z_q \in E$ such that

$$|q(z_q)| > |p(z_q)|. \tag{14}$$

Generalizations of Theorem 2 in the case of a finite set E have been given by Gordon [*2*].

REFERENCES

1. M. FEKETE AND J. L. VON NEUMANN, Über die Lage der Nullstellen gewisser Minimumpolynome. *Jber. Deutsch. Math.-Verein.* **31** (1922), 125–138.
2. Y. GORDON, Properties of generalized juxtapolynomials. *Israel J. Math.* **4** (1966), 177–188.
3. N. N. MEĬMAN, Polynomials deviating least from zero with an arbitrary number of given coefficients. *Soviet Math. Dokl.* **1** (1960), 72–75.
4. T. S. MOTZKIN AND J. L. WALSH, Underpolynomials and infrapolynomials. *Illinois J. Math.* **1** (1957), 406–426.
5. O. SHISHA AND J. L. WALSH, The zeros of infrapolynomials with some prescribed coefficients. *J. Analyse Math.* **9** (1961/62), 111–160.
6. J. L. WALSH, On infrapolynomials with prescribed constant term. *J. Math. Pures Appl.* **37** (1958), 295–316.
7. J. L. WALSH AND M. ZEDEK, On generalized Tchebycheff polynomials. *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 99–104.
8. E. I. ZOLOTAREV, Applications of elliptic functions to questions of functions differing very greatly or very little from zero. *Mem. Acad. Sci., St. Petersburg* **30** (suppl.) No. 5, 1877.